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ON MAXIMAL FAMILIES OF SUBSETS OF A FINITE SET

Hans-Dietrich O. F. GRONAU

Sektion Mathematik, Wilhelm-Pieck-Universität, 25 Rostock, German Democratic Republic

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Let $\mathscr{F}$ be an n-tuple of subsets $X_1, X_2, \ldots, X_n$ of a finite set R of cardinality r. Let us consider 4 conditions which an ordered pair (X, Y) may or may not satisfy, namely

A: if and only if there is $v \in R$: $v \notin X$, $v \in Y$,
B: if and only if there is $v \in R$: $v \in X$, $v \notin Y$,
C: if and only if there is $v \in R$: $v \notin X$, $v \notin Y$,
D: if and only if there is $v \in R$: $v \in X$, $v \in Y$.

Let $\mathsf{P} = \mathsf{P}(\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D})$ be an arbitrary Boolean expression of A, B, C, D. The family $\mathscr{F}$ has the property P, if and only if all ordered pairs (X_i, X_j), $i<j$, of $\mathscr{F}$ satisfy P. Many of the well known conditions for families of subsets of a finite set can be described by certain expressions P. In this paper the maximal cardinality of $\mathscr{F}$ is determined, whenever this exists with exception of one equivalence class.

1. Introduction

Let $\mathscr{F}$ be an n-tuple of subsets $X_1, X_2, \ldots, X_n$ of the interval $R = [1, r]$ of the first r natural numbers, i.e., $\mathscr{F}$ is ordered and can contain some elements manifold. Many authors have determined the maximal value of n if $\mathscr{F}$ satisfies some conditions P. Always they considered $\mathscr{F}$ as an unordered n-tuple of distinct subsets of R. We will enclose these restrictions in P, too. Most of the considered P are of a type which we will characterize in the following way.

Let $v \in R$ Let (X, Y) be an ordered pair of subsets of R. Then exactly one of the four cases holds:

(1) $v \notin X$, $v \in Y$,
(2) $v \in X$, $v \notin Y$,
(3) $v \notin X$, $v \notin Y$,
(4) $v \in X$, $v \in Y$.

Basing on these cases let us consider the following properties, which an ordered pair (X, Y) may or may not possess. (X, Y) has the property

A: if and only if there is $v \in R$: $v \notin X$, $v \in Y$,
B: if and only if there is $v \in R$: $v \in X$, $v \notin Y$,
C: if and only if there is $v \in R$: $v \notin X$, $v \notin Y$,
D: if and only if there is $v \in R$: $v \in X$, $v \in Y$.

Let $\mathsf{P}=\mathsf{P}(\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D})$ be an arbitrary Boolean expression of $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$. $\mathcal{F}$ has the property P if all ordered pairs (X_i, X_j), $1 \leq i < j \leq n$, satisfy the condition P.

If there is a maximal value of n, we will denote this by $n(\mathsf{P}, r)$.

Now we can describe many well known conditions by expressions $\mathsf{P}(\mathsf{A},\mathsf{B},\mathsf{C},\mathsf{D})$.

The following conditions for an ordered pair (X, Y) are equivalent:

$$\begin{aligned}
\mathsf{A}\vee\mathsf{B} &\leftrightarrow X \neq Y,\\
\mathsf{A}\wedge\mathsf{B} &\leftrightarrow X \not\subset Y \not\subset X,\\
\mathsf{C} &\leftrightarrow X \cup Y \neq R,\\
\mathsf{D} &\leftrightarrow X \cap Y \neq \emptyset,\\
\mathsf{A}\wedge\bar{\mathsf{B}} &\leftrightarrow X \subset Y,
\end{aligned}$$

where $\bar{\mathsf{A}}$ denotes *non* A. We will use AB in place of $\mathsf{A}\wedge\mathsf{B}$. If $\mathcal{F}$ has the property AB we call $\mathcal{F}$ a Sperner family (or antichain or clutter) and if $\mathcal{F}$ has the property $\mathsf{A}\bar{\mathsf{B}}$, $\mathcal{F}$ forms a chain.

It is easy to see that $n(\mathsf{A}\vee\mathsf{B}, r)=2^r$ and

$$n((\mathsf{A}\vee\mathsf{B})\mathsf{C}, r)=n((\mathsf{A}\vee\mathsf{B})\mathsf{D}, r)=n(\mathsf{A}\vee\mathsf{B})(\mathsf{C}\vee\mathsf{D}), r)=2^{r-1}.$$

$n((\mathsf{A}\vee\mathsf{B})\mathsf{CD}, r)=2^{r-2}$ was proved by different methods by Anderson [1], Daykin and Lovász [4], Hilton [7], Schönheim [14] and Seymour [16].

The first result of the type $\mathsf{P}=\mathsf{ABP}'(\mathsf{C},\mathsf{D})$ was given by Sperner [17]. He showed

$$n(\mathsf{AB}, r)=\binom{r}{[\frac{1}{2}r]}.^{1}$$

Short proofs were given by Lubell [10], Meshalkin [11] and Yamamoto [18]. Then Milner [12] and later Brace and Daykin [3] proved

$$n(\mathsf{ABC}, r)=n(\mathsf{ABD}, r)=\binom{r}{[\frac{1}{2}(r-1)]}.$$

The latter authors found the relations to $n(\mathsf{ABCD}, r)$ and showed

$$n(\mathsf{ABCD}, r)=\binom{r-1}{[\frac{1}{2}(r-2)]}.$$

This result was given by Katona [9] and Schönheim [15], too. Finally Purdy [13], later Hilton [8] and, independently, Gronau [6] proved by different methods

$$n(\mathsf{AB}(\mathsf{C}\vee\mathsf{D}), r)=\binom{r}{[\frac{1}{2}(r-1)]}.$$

It is worth noting that the condition $\mathsf{AB}(\mathsf{C}\vee\mathsf{D})$ is really weaker than ABC or ABD but the corresponding $n(\mathsf{P}, r)$'s are equal.

For some other P we can find $n(\mathsf{P}, r)$ trivial, e.g. $n(\mathsf{A}\bar{\mathsf{B}}, r)=r+1$.

[1] $[x]$ denotes the greatest integer not exceeding x.

It seems natural to determine $n(\mathsf{P}, r)$ for these considered conditions, but there are many other cases, a few of which may be interesting.

In the present paper we will determine $n(\mathsf{P}, r)$ for arbitrary conditions $\mathsf{P}(\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D})$, whenever $n(\mathsf{P}, r)$ exists. Thus we will be able to say which conditions are equivalent.

2. Main result

We use that every Boolean function $\mathsf{P}(\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}) \neq 0$ can be expressed uniquely by the canonical alternative normal form.

Let $\mathfrak{E}$ be a set of any conjunctions (not necessarily elementary conjunctions). Denote

$$\mathfrak{E}' = \{\mathsf{ABCD}, \mathsf{ABC\bar{D}}, \mathsf{AB\bar{C}D}, \mathsf{AB\bar{C}\bar{D}}\}$$

and denote $\mathfrak{E}'' = \mathfrak{E}' \cup \{\mathsf{A\bar{B}CD}, \mathsf{\bar{A}BCD}\}$. Denote $\mathcal{K}(\mathfrak{E})$ the set of all not empty alternatives of conjunctions of $\mathfrak{E}$, and denote $\mathcal{K}_0(\mathfrak{E}) = \mathcal{K}(\mathfrak{E}) \cup \{\emptyset\}$. CANF(P) denotes the canonical alternative normal form of P. If A' is a conjunction on $\{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}\}$ $\mathsf{A}' \in \text{CANF}(\mathsf{P})$ means that A' is one of the elementary conjunctions of the alternative normal form of P.

We say P is equivalent to P', written $\mathsf{P} \sim \mathsf{P}'$, if and only if $n(\mathsf{P}, r)$ and $n(\mathsf{P}', r)$ exist and $n(\mathsf{P}, r) = n(\mathsf{P}', r)$ holds for all $r \geq 2$. Obviously, this is an equivalence relation.

We define 48 classes $\mathcal{K}_{i,j}$ of expressions (for better systematising we define 48 classes, although only 42 classes are different) by $\mathsf{P} \in \mathcal{K}_{i,j} \leftrightarrow n(\mathsf{P}, r)$ has the value described in Table 1 for $r \geq 2$, where $i \in \{1, 2, \ldots, 16\}$ and $j \in \{1, 2, 3\}$ and $f(r) = n(\mathsf{ACD} \vee \mathsf{AB\bar{C}\bar{D}}, r)$.

Now we are able to formulate our main result.

Theorem 1. *Every expression* $\mathsf{P}(\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D})$ *for which* $n(\mathsf{P}, r)$ *exists lies in one of the classes* $\mathcal{K}_{i,j}$ *as described in Table* 1, *and no class is empty.*

The statement of Theorem 1 means that the classes $\mathcal{K}_{i,j}$ are the equivalence classes of our relation.

Notice that 2^{2^4} Boolean functions of 4 variables exist.

In Section 3 we will show that for exactly 2^{13} functions P $n(\mathsf{P}, r)$ exists, and we will determine these functions. In Section 4 we will see that every function P tells us whether $\emptyset \in \mathcal{F}$ and/or $R \in \mathcal{F}$ hold or not, where $\mathcal{F}$ is a family with maximal cardinality $n(\mathsf{P}, r)$. So we will find for every P with $\emptyset \in \mathcal{F}$ or $R \in \mathcal{F}$ a function P' with a family $\mathcal{F}'$ satisfying $\mathcal{F}' = \mathcal{F} \setminus \{\emptyset\} \setminus \{R\}$. Hence we only need to consider the 63 functions P with $\mathsf{P} \neq 0$, $\emptyset$, $R \notin \mathcal{F}$. In Section 5 we will give some lemmata that will show that certain functions are equivalent. Finally in Section 6 we will

Table 1

$i \backslash j$	1	2	3
1	2^r-2	2^r-1	2^r
2	$2^{r-1}-1$	2^{r-1}	$2^{r-1}+1$
3	2^{r-2}	$2^{r-2}+1$	$2^{r-2}+2$
4	$2^{r-1}+\binom{r-1}{[\frac{1}{2}(r-2)]}-1$	$2^{r-1}+\binom{r-1}{[\frac{1}{2}(r-2)]}$	$2^{r-1}+\binom{r-1}{[\frac{1}{2}(r-2)]}+1$
5	$f(r)$	$f(r)+1$	$f(r)+2$
6	$\binom{r}{[\frac{1}{2}r]}$	$\binom{r}{[\frac{1}{2}r]}+1$	$\binom{r}{[\frac{1}{2}r]}+2$
7	$\binom{r}{[\frac{1}{2}(r-1)]}$	$\binom{r}{[\frac{1}{2}(r-1)]}+1$	$\binom{r}{[\frac{1}{2}(r-1)]}+2$
8	$\binom{r-1}{[\frac{1}{2}(r-2)]}$	$\binom{r-1}{[\frac{1}{2}(r-2)]}+1$	$\binom{r-1}{[\frac{1}{2}(r-2)]}+2$
9	$2\binom{r-1}{[\frac{1}{2}(r-2)]}$	$2\binom{r-1}{[\frac{1}{2}(r-2)]}+1$	$2\binom{r-1}{[\frac{1}{2}(r-2)]}+2$
10	$4r-6$	$4r-5$	$4r-4$
11	$2r-2$	$2r-1$	$2r$
12	$2r-3$	$2r-2$	$2r-1$
13	r	$r+1$	$r+2$
14	$r-1$	r	$r+1$
15	2	3	4
16	1	2	3

determine $n(\mathsf{P}, r)$ for the remaining 23 functions and thus show that every one of these functions lies in one of the 16 classes $\mathcal{K}_{i,1}$ $(i = 1, 2, \ldots, 16)$.

3. Existence of $n(\mathsf{P}, r)$

We suppose that no pair (X, Y) satisfies $\mathsf{P} \equiv 0$, i.e. $0 \in \mathcal{K}_{16,1}$.

Lemma 1. (1) $\bar{\mathsf{A}}\bar{\mathsf{B}}\bar{\mathsf{C}}\bar{\mathsf{D}} \in \mathcal{K}_{16,1}$.
(2) *If* $\bar{\mathsf{A}}\bar{\mathsf{B}}\bar{\mathsf{C}}\bar{\mathsf{D}} \notin \mathrm{CANF}(\mathsf{P})$, *then* $\mathsf{P} \sim \mathsf{P} \vee \bar{\mathsf{A}}\bar{\mathsf{B}}\bar{\mathsf{C}}\bar{\mathsf{D}}$.

Proof. (1) An arbitrary pair (X, Y) of subsets of R satisfies $\mathsf{A} \vee \mathsf{B} \vee \mathsf{C} \vee \mathsf{D}$, i.e. no pair can satisfy $\bar{\mathsf{A}}\bar{\mathsf{B}}\bar{\mathsf{C}}\bar{\mathsf{D}}$. Hence $\mathcal{F}$ cannot contain a pair, i.e. $n \leqslant 1$. If $|\mathcal{F}| = 1$, $\bar{\mathsf{A}}\bar{\mathsf{B}}\bar{\mathsf{C}}\bar{\mathsf{D}}$ is satisfied. (2) follows by (1) immediately.

Theorem 2. $n(\mathsf{P}, r)$ *does not exist if and only if*

(1) $\bar{\mathsf{A}}\bar{\mathsf{B}}\mathsf{C}\mathsf{D} \in \mathrm{CANF}(\mathsf{P})$ *or*
(2) $\bar{\mathsf{A}}\bar{\mathsf{B}}\bar{\mathsf{C}}\mathsf{D} \in \mathrm{CANF}(\mathsf{P})$ *or*
(3) $\bar{\mathsf{A}}\bar{\mathsf{B}}\mathsf{C}\bar{\mathsf{D}} \in \mathrm{CANF}(\mathsf{P})$.

Proof. If one of these 3 cases holds, $\mathcal{F}$ can contain one element unlimitedly often, e.g. the sets $\{1\}$ (1), R (2) or $\emptyset$ (3). On the other hand, all functions ($\neq 0$) not described in Theorem 2 are equivalent to a function P' (Lemma 1) having a nonempty CANF(P') with $\mathsf{\bar{A}\bar{B}CD}$, $\mathsf{\bar{A}\bar{B}\bar{C}D}$, $\mathsf{\bar{A}\bar{B}C\bar{D}}$, $\mathsf{\bar{A}\bar{B}\bar{C}\bar{D}} \notin \mathrm{CANF}(\mathsf{P}')$, i.e. all conjunctions contained in CANF(P') contain at least A or B. So the elements of $\mathcal{F}$ must be distinct, and 2^r is an upper bound for $|\mathcal{F}|$. Hence $n(\mathsf{P}, r)$ exists.

An immediate consequence from Theorem 2 is that there are exactly $2^{13} = 8192$ functions P where $n(\mathsf{P}, r)$ exists.

4. A theorem on reducing

In all that follows let P be an arbitrary Boolean function with $\mathsf{P} \not\equiv 0$ and $\mathsf{\bar{A}\bar{B}CD}$, $\mathsf{\bar{A}\bar{B}\bar{C}D}$, $\mathsf{\bar{A}\bar{B}C\bar{D}}$, $\mathsf{\bar{A}\bar{B}\bar{C}\bar{D}} \notin \mathrm{CANF}(\mathsf{P})$.

The next three lemmata are easy to verify.

Lemma 2. *An ordered pair (X, Y) of subsets of R satisfies* $\mathsf{A\bar{B}C\bar{D}}$, $\mathsf{A\bar{B}\bar{C}D}$, $\mathsf{\bar{A}BC\bar{D}}$, *or* $\mathsf{\bar{A}B\bar{C}D}$ *if and only if* $(X = \emptyset;\ Y \neq \emptyset, R)$, $(X \neq \emptyset, R;\ Y = R)$, $(X \neq \emptyset, R;\ Y = \emptyset)$ *or* $(X = R;\ Y \neq \emptyset, R)$, *respectively.*

Lemma 3. *An ordered pair (X, Y) of subsets of R satisfies* $\mathsf{A\bar{B}\bar{C}\bar{D}}$ *or* $\mathsf{\bar{A}B\bar{C}\bar{D}}$ *if and only if* $(X = \emptyset;\ Y = R)$ *or* $(X = R;\ Y = \emptyset)$, *respectively.*

Lemma 4. *Let $\mathcal{F}$ satisfy* P *with* $\mathrm{CANF}(\mathsf{P}) \in \mathcal{K}(\mathfrak{E}'')$. *Then* $\emptyset \notin \mathcal{F}$ *and* $R \notin \mathcal{F}$.

Theorem 3. *Let* $\mathsf{P} = \mathsf{P}' \vee \mathsf{F}''$, *where* $\mathrm{CANF}(\mathsf{P}') \in \mathcal{K}_0(\mathfrak{E}'')$ *and*

$$\mathrm{CANF}(\mathsf{P}'') \in \mathcal{K}(\{\mathsf{A\bar{B}C\bar{D}}, \mathsf{A\bar{B}\bar{C}D}, \mathsf{\bar{A}B\bar{C}D}, \mathsf{\bar{A}BC\bar{D}}, \mathsf{A\bar{B}\bar{C}\bar{D}}, \mathsf{\bar{A}B\bar{C}\bar{D}}\}).$$

Then

$$n(\mathsf{P}, r) = \begin{cases} n(\mathsf{P}', r) + 2, & \textit{if } \mathsf{A\bar{B}C\bar{D}}, \mathsf{\bar{A}B\bar{C}D}, \mathsf{A\bar{B}\bar{C}\bar{D}} \in \mathrm{CANF}(\mathsf{P}'') \textit{ or} \\ & \quad \mathsf{A\bar{B}C\bar{D}}, \mathsf{\bar{A}B\bar{C}D}, \mathsf{\bar{A}B\bar{C}\bar{D}} \in \mathrm{CANF}(\mathsf{P}'') \textit{ or} \\ & \quad \mathsf{A\bar{B}C\bar{D}}, \mathsf{A\bar{B}\bar{C}D}, \mathsf{A\bar{B}\bar{C}\bar{D}} \in \mathrm{CANF}(\mathsf{P}'') \textit{ or} \\ & \quad \mathsf{\bar{A}BC\bar{D}}, \mathsf{\bar{A}B\bar{C}D}, \mathsf{\bar{A}B\bar{C}\bar{D}} \in \mathrm{CANF}(\mathsf{P}'') \textit{ or} \\ & \quad \mathsf{\bar{A}BC\bar{D}}, \mathsf{A\bar{B}\bar{C}D}, \mathsf{A\bar{B}\bar{C}\bar{D}} \in \mathrm{CANF}(\mathsf{P}'') \textit{ or} \\ & \quad \mathsf{\bar{A}BC\bar{D}}, \mathsf{A\bar{B}\bar{C}D}, \mathsf{\bar{A}B\bar{C}\bar{D}} \in \mathrm{CANF}(\mathsf{P}''), \\ n(\mathsf{P}', r), & \textit{if } \mathsf{P}' \not\equiv 0 \textit{ and} \\ & \quad \mathrm{CANF}(\mathsf{P}'') \in \mathcal{K}(\{\mathsf{A\bar{B}\bar{C}\bar{D}}, \mathsf{\bar{A}B\bar{C}\bar{D}}\}), \\ 2, & \textit{if } \mathsf{P}' \equiv 0 \textit{ and} \\ & \quad \mathrm{CANF}(\mathsf{P}'') \in \mathcal{K}(\{\mathsf{A\bar{B}\bar{C}\bar{D}}, \mathsf{\bar{A}B\bar{C}\bar{D}}\}), \\ n(\mathsf{P}', r) + 1, & \textit{otherwise.} \end{cases}$$

Proof. Let $\mathscr{F}$ be a family satisfying $\mathsf{P}=\mathsf{P}'\vee\mathsf{P}''$ with $|\mathscr{F}|=n(\mathsf{P},r)$. Let P' and P'' be defined as in Theorem 3. By the Lemmata 2, 3, and 4 we obtain that $\mathscr{F}\setminus\{\emptyset\}\setminus\{R\}$ satisfies P' and $|\mathscr{F}\setminus\{\emptyset\}\setminus\{R\}|=n(\mathsf{P}',r)$ holds. Hence

$$n(\mathsf{P}',r)\leq n(\mathsf{P},r)\leq n(\mathsf{P}',r)+2. \tag{1}$$

1. Let equality hold in the right hand inequality. Then $\emptyset$, $R\in\mathscr{F}$ and $\mathsf{A\bar{B}\bar{C}\bar{D}}\in$ CANF(P'') or $\mathsf{\bar{A}B\bar{C}\bar{D}}\in$ CANF(P''). Furthermore, $n(\mathsf{P}',r)\geq 1$. Thus there is at least one element $X\in\mathscr{F}$ with $X\notin\{\emptyset,R\}$. In $\mathscr{F}$ exists at least one ordered pair $(\emptyset,X)$ or $(X,\emptyset)$ and one pair (R,X) or (X,R). Hence $\mathsf{A\bar{B}C\bar{D}}$ or $\mathsf{\bar{A}BC\bar{D}}$ lies in CANF(P'') as well as $\mathsf{A\bar{B}\bar{C}D}$ or $\mathsf{\bar{A}B\bar{C}D}$. Now there are 8 possible combinations of these containing conjunctions, the 6 described in Theorem 3 and $\mathsf{A\bar{B}C\bar{D}}$, $\mathsf{A\bar{B}\bar{C}D}$, $\mathsf{\bar{A}BC\bar{D}}$ and $\mathsf{\bar{A}BC\bar{D}}$, $\mathsf{\bar{A}B\bar{C}D}$, $\mathsf{A\bar{B}\bar{C}\bar{D}}$. It is easy to verify that the 6 combinations described in Theorem 3 are sufficient for $\emptyset,R\in F$. The last two are not, e.g. $\mathsf{A\bar{B}C\bar{D}}$, $\mathsf{A\bar{B}\bar{C}D}$, $\mathsf{\bar{A}B\bar{C}\bar{D}}$ implies the existence of the following ordered pairs in $\mathscr{F}$: $(\emptyset,X)$, (X,R) and $(R,\emptyset)$, which cannot be true, because each element of $\mathscr{F}$ exists only one time.

2. Let equality hold in the left hand inequality. Let $\mathscr{F}'$ denote $\mathscr{F}\setminus\{\emptyset\}\setminus\{R\}$. Then $\mathscr{F}'$ has maximal cardinality under the condition P'. If $\mathsf{A\bar{B}C\bar{D}}$, or $\mathsf{A\bar{B}\bar{C}D}$, or $\mathsf{\bar{A}B\bar{C}D}$, or $\mathsf{\bar{A}BC\bar{D}}$ is an element of CANF(P'') we construct a family $\mathscr{G}=(\emptyset,\mathscr{F}')$, or $(\mathscr{F}',R)$, or $(R,\mathscr{F}')$, or $(\mathscr{F}',\emptyset)$, respectively, where $(\emptyset,\mathscr{F}')$ means an $(|\mathscr{F}'|+1)$-tuple. Clearly, $\mathscr{G}$ satisfies P in every case. Hence $n(\mathsf{P}',r)=|\mathscr{F}'|<|\mathscr{G}|=n(\mathsf{P},r)$, which contradicts our supposition. Hence CANF(P'')$\in\mathscr{K}(\{\mathsf{A\bar{B}\bar{C}\bar{D}},\mathsf{\bar{A}B\bar{C}\bar{D}}\})$. By Lemmata 3 and 4 it follows that $\mathscr{F}$ satisfies P' or $\mathscr{F}$ satisfies P''. Thus $n(\mathsf{P},r)=\max(n(\mathsf{P}'',r), n(\mathsf{P}',r))$. CANF($\mathsf{P}''$) is not empty. Hence $n(\mathsf{P}'',r)=2$. If $\mathsf{P}'\not\equiv 0$ there is at least one pair (X,Y) satisfying P', i.e. $n(\mathsf{P}',r)\geq 2$ and $n(\mathsf{P},r)=n(\mathsf{P}',r)$. Finally we notice that $n(0,r)=1$, which we have supposed. Hence $n(\mathsf{P},r)=n(\mathsf{P}'',r)=2$ if $\mathsf{P}'\equiv 0$.

In other cases equality cannot hold in (1). This completes the proof of Theorem 3.

Using Theorem 3 we only need to consider P' with CANF(P')$\in\mathscr{K}(\mathfrak{S}'')$. If we have shown that $\mathsf{P}'\in\mathscr{K}_{i,1}$, we know in which of the classes $\mathscr{K}_{i,j}$ ($j\in\{1,2,3\}$) P lies.

5. Some lemmata on equivalent functions

Lemma 5. (1) $\mathsf{P(A,B,C,D)}\sim\mathsf{P(B,A,C,D)}$,
(2) $\mathsf{P(A,B,C,D)}\sim\mathsf{P(A,B,D,C)}$.

Proof. (1) $\mathscr{F}=(X_1,X_2,\ldots,X_n)$ satisfies $\mathsf{P(A,B,C,D)}$ if and only if $\mathscr{F}'=(X_n,\ldots,X_2,X_1)$ satisfies $\mathsf{P(B,A,C,D)}$.

(2) $\mathscr{F}$ satisfies $\mathsf{P(A,B,C,D)}$ if and only if $\mathscr{F}''=(R\setminus X_1,R\setminus X_2,\ldots,R\setminus X_n)$ satisfies $\mathsf{P(B,A,D,C)}$. (1) completes the proof.

Let $\mathscr{Q}$ be an operator which orders the elements of $\mathscr{F}$ in such a way that $|Y_i|\leq|Y_j|$ for $1\leq i<j\leq n$ and $\mathscr{Q}(\mathscr{F})=(Y_1,\ldots,Y_n)$.

Lemma 6. *Let* P *and* P' *be arbitrary Boolean functions in two variables. Then*

(1) $(\mathsf{A}\bar{\mathsf{B}} \vee \bar{\mathsf{A}}\mathsf{B})\mathsf{P}(\mathsf{C},\mathsf{D}) \sim \mathsf{A}\bar{\mathsf{B}}\mathsf{P}(\mathsf{C},\mathsf{D})$, *and*

(2) $(\mathsf{A} \vee \mathsf{B})\mathsf{P}'(\mathsf{C},\mathsf{D}) \sim \mathsf{A}\mathsf{P}'(\mathsf{C},\mathsf{D})$.

Proof. The conditions on the right hand sides are sharper than the conditions on the left hand sides. So we only have to show that for a maximal family $\mathscr{F}$ satisfying the left hand condition there is a family $\mathscr{F}'$ of the same cardinality satisfying the right hand condition. Obviously, $|\mathscr{Q}(\mathscr{F})| = |\mathscr{F}|$ holds and if (X, Y) is an arbitrary ordered pair of $\mathscr{F}$ then these elements belong to $\mathscr{Q}(\mathscr{F})$ too. They form an ordered pair (Z, W) with either $(Z, W) = (X, Y)$ or $(Z, W) = (Y, X)$. Notice that the pairs of $\mathscr{Q}(\mathscr{F})$ satisfy A always, and if (X, Y) satisfies $\mathsf{P}(\mathsf{C},\mathsf{D})$ then (Z, W) satisfies $\mathsf{P}(\mathsf{C},\mathsf{D})$ too. If (X, Y) satisfies $\mathsf{ABP}(\mathsf{C},\mathsf{D})$, $\mathsf{A}\bar{\mathsf{B}}\mathsf{P}(\mathsf{C},\mathsf{D})$ or $\bar{\mathsf{A}}\mathsf{BP}(\mathsf{C},\mathsf{D})$, we obtain that (Z, W) satisfies $\mathsf{ABP}(\mathsf{C},\mathsf{D})$, $\mathsf{A}\bar{\mathsf{B}}\mathsf{P}(\mathsf{C},\mathsf{D})$ or $\mathsf{A}\bar{\mathsf{B}}\mathsf{P}(\mathsf{C},\mathsf{D})$, respectively. Notice that $\mathsf{A} \vee \mathsf{B} = \mathsf{AB} \vee \mathsf{A}\bar{\mathsf{B}} \vee \bar{\mathsf{A}}\mathsf{B}$. Hence the lemma follows immediately.

The next two lemmata are immediate consequences from Lemmata 5 and 6.

Lemma 7. *Let* $\mathsf{P} \in \mathscr{K}_0(\mathfrak{E}')$. *Then*

(1) $\mathsf{P} \vee \mathsf{A}\bar{\mathsf{B}}\mathsf{CD} \vee \bar{\mathsf{A}}\mathsf{BCD} \sim \mathsf{P} \vee \mathsf{A}\bar{\mathsf{B}}\mathsf{CD}$,

(2) $\mathsf{P} \vee \bar{\mathsf{A}}\mathsf{BCD} \sim \mathsf{P} \vee \mathsf{A}\bar{\mathsf{B}}\mathsf{CD}$.

Lemma 8. *Let* $\mathsf{P} \in \mathscr{K}_0(\{\mathsf{ABCD}, \mathsf{AB}\bar{\mathsf{C}}\bar{\mathsf{D}}, \mathsf{A}\bar{\mathsf{B}}\mathsf{CD}\})$. *Then* $\mathsf{P} \vee \mathsf{AB}\bar{\mathsf{C}}\mathsf{D} \sim \mathsf{P} \vee \mathsf{ABC}\bar{\mathsf{D}}$.

6. $n(\mathsf{P}, r)$ of the reduced conditions P

By the lemmata of Section 5 we can reduce the conditions to be considered to the following 23 types (we may omit $\bar{\mathsf{A}}\mathsf{BCD}$ in CANF(P) by Lemma 7 and exchange $\mathsf{AB}\bar{\mathsf{C}}\mathsf{D}$ for $\mathsf{ABC}\bar{\mathsf{D}}$, if $\mathsf{AB}\bar{\mathsf{C}}\mathsf{D} \in$ CANF(P) and $\mathsf{ABC}\bar{\mathsf{D}} \notin$ CANF(P), by Lemma 8), where for simplicity the well known relations $\mathsf{M} \vee \mathsf{MN} = \mathsf{M}$ and $\mathsf{MN} \vee \mathsf{M}\bar{\mathsf{N}} = \mathsf{M}$ were used as often as possible.

The first 5 results are described in Section 1. In general we prove an upper bound for $n(\mathsf{P}, r)$ in the other cases. This being also a lower bound follows by the given examples immediately.

1. $\mathsf{ACD} \in \mathscr{K}_{3,1}$.

2. $\mathsf{AB} \in \mathscr{K}_{6,1}$.

3. $\mathsf{ABC} \in \mathscr{K}_{7,1}$.

4. $\mathsf{ABCD} \in \mathscr{K}_{8,1}$.

5. $\mathsf{ABC} \vee \mathsf{ABD} \in \mathscr{K}_{7,1}$.

6. $\mathsf{ABC} \vee \mathsf{ACD} \vee \mathsf{AB}\bar{\mathsf{D}} \in \mathscr{K}_{4,1}$. If $\mathscr{F}$ satisfies this condition, $\mathscr{F}$ satisfies the following condition too. With an arbitrary element $X \in \mathscr{F}$, $\mathscr{F}$ contains at most one element Y with X and Y having no empty intersection. Erdös [5] determined the maximal cardinality of $\mathscr{F}$ in the dual case. Example 1 (see the list of examples, Table 2).

Table 2. List of examples

No.	Example for $\mathscr{F}=(X_1, X_2, \ldots, X_n)$
1	$\mathscr{Q}(\mathscr{G})$ with $\mathscr{G}=\{X: X\subset R, 1\leqslant\lvert X\rvert\leqslant\frac{1}{2}r\}$ if r even, and $\mathscr{G}=\{X: X\subset R, 1\leqslant\lvert X\rvert\leqslant\frac{1}{2}(r-1)$ *or* $(\lvert X\rvert=\frac{1}{2}(r+1)$ and $r\notin X)\}$ if r odd.
2	$X_1=\{1\}, X_2=R\setminus\{1\}$.
3	$X_i=\bigcup_{j=1}^{i}\{j\}$ $(i=1,2,\ldots,r-1)$.
4	$X_i=\{i\}$ $(i=1,2,\ldots,r)$.
5	$\mathscr{Q}(\mathscr{G})$ with $\mathscr{G}$ consists of all subsets of $R\setminus\{1\}$ without $\emptyset$.
6	$\mathscr{F}$ consists of all subsets of R of cardinality $[\frac{1}{2}r]$.
7	r odd: $\mathscr{F}$ consists of all subsets of R of cardinality $\frac{1}{2}(r-1)$ containing the element 1 and all subsets of R of cardinality $\frac{1}{2}(r+1)$ not containing the element 1. r even: $\mathscr{F}$ consists of all subsets of R of cardinality $\frac{1}{2}r$.
8	$\mathscr{Q}(\mathscr{G})$, where $\mathscr{G}$ consists of all sets defined in Examples 3 and 4.
9	$\mathscr{F}$ consists of all sets defined in Example 8 without $\{1,\ldots,r-1\}$.
10	$\mathscr{Q}(\mathscr{G})$, where $\mathscr{G}$ consists of all sets defined in Example 9 and its complements.

7. $\mathbf{AB\bar{C}\bar{D}}\in\mathscr{K}_{15,1}$. $\mathscr{F}$ can contain at most X and $R\setminus X$. Example 2.
8. $\mathbf{A\bar{B}CD}\in\mathscr{K}_{14,1}$. $\mathscr{F}$ forms a chain, without $\emptyset$ and R, i.e. $n\leqslant r-1$. Example 3.
9. $\mathbf{A\bar{B}CD}\vee\mathbf{AB\bar{C}\bar{D}}\in\mathscr{K}_{14,1}$. Let (X, Y) be a pair of $\mathscr{F}$ satisfying $\mathbf{AB\bar{C}\bar{D}}$, i.e. $Y=R\setminus X$. Let $Z\in\mathscr{F}\setminus\{X\}\setminus\{Y\}$. If (Z, X) and (Z, Y) lie in $\mathscr{F}$, both pairs satisfy $\mathbf{A\bar{B}CD}$. Hence $Z\subset X$ and $Z\subset R\setminus X$ hold and it follows $Z=\emptyset$, which contradicts our supposition that $\emptyset, R\in\mathscr{F}$. In analogy follows that it is impossible that (X, Z) and (Z, Y), or (X, Z) and (Y, Z), lie in $\mathscr{F}$, i.e. $\mathscr{F}$ satisfies $\mathbf{A\bar{B}CD}$ or $\mathscr{F}$ satisfies $\mathbf{AB\bar{C}\bar{D}}$. By **7** and **8** we obtain

$$n(\mathbf{A\bar{B}CD}\vee\mathbf{AB\bar{C}\bar{D}}, r)=\max(n(\mathbf{A\bar{B}CD}, r), n(\mathbf{AB\bar{C}\bar{D}}, r))=\begin{cases} r-1 & \text{if } r\geqslant 3,\\ 2 & \text{if } r=2.\end{cases}$$

(In spite of the case $r=2$ we put this condition in $\mathscr{K}_{14,1}$.)
10. $\mathbf{AB\bar{D}}\in\mathscr{K}_{13,1}$. By $\mathbf{\bar{D}}$ every element of R can lie at most in one X_i, i.e. $n\leqslant r$. Example 4.
11. $\mathbf{ABC\bar{D}}\in\mathscr{K}_{13,1}$. By **10** we get $n\leqslant r$. Example 4 for $r\geqslant 3$. For $r=2$ we obtain $n=1$.
12. $\mathbf{AB\bar{C}}\vee\mathbf{AB\bar{D}}\in\mathscr{K}_{13,1}$. Consider 3 subsets X, Y, Z of R, where (X, Y) satisfies $\mathbf{AB\bar{C}}$ and (Y, Z) satisfies $\mathbf{AB\bar{D}}$. Then for all $v\in R$: $v\notin X$ implies $v\in Y$ and further $v\notin Z$, i.e. (X, Z) does not satisfy $\mathbf{A}$ which contradicts the property of $\mathscr{F}$. Analogously (X, Y) cannot satisfy $\mathbf{AB\bar{D}}$ if (Y, Z) satisfies $\mathbf{AB\bar{C}}$, etc. Hence we obtain that $\mathscr{F}$ satisfies $\mathbf{AB\bar{D}}$ or $\mathscr{F}$ satisfies $\mathbf{AB\bar{C}}$. By $\mathbf{AB\bar{C}}\sim\mathbf{AB\bar{D}}$ we obtain $\mathbf{AB\bar{C}}\vee\mathbf{AB\bar{D}}\sim\mathbf{AB\bar{D}}$.
13. $\mathbf{ABC\bar{D}}\vee\mathbf{AB\bar{C}D}\in\mathscr{K}_{13,1}$. We get $r=n(\mathbf{ABC\bar{D}}, r)\leqslant n(\mathbf{ABC\bar{D}}\vee\mathbf{AB\bar{C}D}, r)\leqslant n(\mathbf{AB\bar{C}}\vee\mathbf{AB\bar{D}}, r)=r$ by **11** and **12** for $r\geqslant 3$. For $r=2$ it follows $n=1$ immediately.
14. $\mathbf{ABC}\vee\mathbf{ABD}\vee\mathbf{ACD}\in\mathscr{K}_{2,1}$. $\mathscr{F}$ contains no set and its complement. Example 5.
15. $\mathbf{ABC}\vee\mathbf{ACD}\in\mathscr{K}_{2,1}$. This follows by **14** and Example 5.

16. $\mathbf{AB} \vee \mathbf{ACD} \in \mathcal{K}_{1,1}$. Arbitrary pairs (X, Y) with $X, Y \notin \{\emptyset, R\}$ satisfy this condition.
17. $\mathbf{ABC} \vee \mathbf{AB\bar{D}} \in \mathcal{K}_{6,1}$. Clearly, $n(\mathbf{ABC} \vee \mathbf{AB\bar{D}}, r) \leqslant n(\mathbf{AB}, r)$ holds. Example 6.
18. $\mathbf{ABCD} \vee \mathbf{AB\bar{C}\bar{D}} \in \mathcal{K}_{9,1}$. We can divide $\mathcal{F}$ in $\mathcal{F}_1$ and $\mathcal{F}_2$, where $\mathcal{F}_2$ consists only of complementary sets of $\mathcal{F}_1$, and $\mathcal{F}_1$ do not contain a set and its complement. Hence $\mathcal{F}_1$ as well as $\mathcal{F}_2$ satisfy $\mathbf{ABCD}$, i.e. $n \leqslant 2\binom{r-1}{[\frac{1}{2}(r-2)]}$ by **4**. Example 7. This maximum holds exactly if $\mathcal{F}$ consists of pairs of complementary subsets. The maximum of $\mathcal{F}$ in this sense was also determined by Bollobás [2].
19. $\mathbf{AB\bar{D}} \vee \mathbf{A\bar{B}CD} \in \mathcal{K}_{11,1}$. Let (X, Y) be an arbitrary pair of $\mathcal{F}$. Then $X \subset Y$ (we use the inclusion in the strong sense) or $X \cap Y = \emptyset$ holds. Hence there is uniquely a set $\mathcal{G}(\mathcal{F}) \subseteqq \mathcal{F}$ with

(1) $X \cap Y = \emptyset$ for all pairs (X, Y) with $X, Y \in \mathcal{G}(\mathcal{F})$,

(2) for all $X \in \mathcal{F} \setminus \mathcal{G}(\mathcal{F})$ there is an element $Y \in \mathcal{G}(\mathcal{F})$ with $X \subset Y$.

Let $X \in \mathcal{G}(\mathcal{F})$ and let $\mathcal{X} = \{Y\colon Y \in \mathcal{F}, Y \subset X\}$. Then

$$\mathcal{F} = \mathcal{G}(\mathcal{F}) \cup \bigcup_{X \in \mathcal{G}} \mathcal{X}.$$

The elements of $\mathcal{X}$ satisfy $\mathbf{AB\bar{D}} \vee \mathbf{A\bar{B}CD}$ too, i.e. $|\mathcal{X}| \leqslant n(\mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{D}}, |X|)$. Let $\mathcal{F}$ be an arbitrary family on R satisfying $\mathbf{AB\bar{D}} \vee \mathbf{A\bar{B}CD}$. Then

$$|\mathcal{F}| \leqslant |\mathcal{G}(\mathcal{F})| + \sum_{X \in \mathcal{G}} n(\mathbf{AB\bar{D}} \vee \mathbf{A\bar{B}CD}, |X|). \tag{2}$$

We will prove $n(\mathbf{AB\bar{D}} \vee \mathbf{A\bar{B}CD}, r) = 2r - 2$ by induction on r.

(1) $r = 1$. (We supposed at the beginning $r \geqslant 2$, but if there is $X \in \mathcal{G}(\mathcal{F})$ with $|X| = 1$, $\mathcal{X}$ forms a family on an one-element set satisfying $\mathbf{AB\bar{D}} \vee \mathbf{A\bar{B}CD}$. This case can happen by induction.) Clearly, $n = 0$.
$r = 2$. The only pair $(\{1\}, \{2\})$ satisfies $\mathbf{AB\bar{D}}$.

(2) Let $\mathcal{F}$ be an arbitrary family on R satisfying $\mathbf{AB\bar{D}} \vee \mathbf{A\bar{B}CD}$. We distinguish two cases.

(2.1) $|\mathcal{G}(\mathcal{F})| = 1$. By $X \subset R$ we have $|X| \leqslant r - 1$. Hence

$$|\mathcal{F}| \leqslant 1 + n(\mathbf{AB\bar{D}} \vee \mathbf{A\bar{B}CD}, r - 1) = 2r - 3 \quad \text{by (2)}.$$

(2.2) $|\mathcal{G}(\mathcal{F})| \geqslant 2$. Then $\sum_{X \in \mathcal{G}} |X| \leqslant r$ and we obtain

$$\begin{aligned} |\mathcal{F}| &\leqslant |\mathcal{G}(\mathcal{F})| + \sum_{X \in \mathcal{G}} (2|X| - 2) \\ &= 2 \sum_{X \in \mathcal{G}} |X| - |\mathcal{G}(\mathcal{F})| \\ &\leqslant 2r - 2 \quad \text{by (2)}. \end{aligned}$$

Example 8 completes the proof.
20. $\mathbf{ABC\bar{D}} \vee \mathbf{A\bar{B}CD} \in \mathcal{K}_{12,1}$. Let (X, Y) be an arbitrary pair of $\mathcal{F}$. Then $X \subset Y$ or $(X \cap Y = \emptyset$ and $X \cup Y \neq R)$. There is a set $\mathcal{G}(\mathcal{F}) \subseteqq \mathcal{F}$ defined as in **19**. We notice

$X \in \mathscr{G}(\mathscr{F})$ implies that $\mathscr{X}$ satisfies $\mathbf{AB\bar{D}} \vee \mathbf{A\bar{B}CD}$, but not necessarily $\mathbf{ABC\bar{D}} \vee \mathbf{A\bar{B}CD}$. Hence for arbitrary families $\mathscr{F}$ satisfying $\mathbf{ABC\bar{D}} \vee \mathbf{A\bar{B}CD}$ we have

$$|\mathscr{F}| \leqslant |\mathscr{G}(\mathscr{F})| + \sum_{X \in \mathscr{G}} n(\mathbf{AB\bar{D}} \vee \mathbf{A\bar{B}CD}, |X|). \tag{3}$$

Using (3) we will prove $n(\mathbf{ABC\bar{D}} \vee \mathbf{A\bar{B}CD}, r) = 2r-3$ by induction on r.

(1) $r=2$. Then the only possible pair $(\{1\},\{2\})$ does not satisfy $\mathbf{ABC\bar{D}} \vee \mathbf{A\bar{B}CD}$, i.e. $n(\mathbf{ABC\bar{D}} \vee \mathbf{A\bar{B}CD}, 2) = 1$.

(2) Let $\mathscr{F}$ be an arbitrary family on R satisfying $\mathbf{ABC\bar{D}} \vee \mathbf{A\bar{B}CD}$. We will use (3) and **19**. We distinguish three cases.

(2.1) $|\mathscr{G}(\mathscr{F})| = 1$. By $X \subset R$ we have $|X| \leqslant r-1$. Hence

$$|\mathscr{F}| \leqslant 1 + 2(r-1) - 2 = 2r-3.$$

(2.2) $|\mathscr{G}(\mathscr{F})| = 2$. Let $\mathscr{G}(\mathscr{F}) = (X_1, X_2)$. Then $|X_1| + |X_2| \leqslant r-1$, since $|X_1| + |X_2| = r$ implies $X_1 = R \setminus X_2$, but then (X_1, X_2) does not satisfy $\mathbf{ABC\bar{D}} \vee \mathbf{A\bar{B}CD}$. Hence

$$|\mathscr{F}| \leqslant 2 + (2|X_1| - 2) + (2|X_2| - 2) \leqslant 2r-4.$$

(2.3) $|\mathscr{G}(\mathscr{F})| \geqslant 3$. Then $\sum_{X \in \mathscr{G}} |X| \leqslant r$ and we obtain

$$|\mathscr{F}| \leqslant |\mathscr{G}(\mathscr{F})| + \sum_{X \in \mathscr{G}} (2|X| - 2) \leqslant 2r - |\mathscr{G}(\mathscr{F})| \leqslant 2r-3.$$

Example 9 completes the proof.

21. $\mathbf{ABC\bar{D}} \vee \mathbf{AB\bar{C}D} \vee \mathbf{A\bar{B}CD} \in \mathscr{K}_{12,1}$. $\mathscr{F}$ contains no set and its complement. We form a family $\mathscr{F}'$ of cardinality $|\mathscr{F}|$ by the following process:

(1) If $1 \in X \in \mathscr{F}$, then $R \setminus X \in \mathscr{F}'$.

(2) If $1 \notin X \in \mathscr{F}$, then $X \in \mathscr{F}'$.

The pairs of $\mathscr{F}'$ satisfy $\mathbf{\bar{A}BCD} \vee \mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{C}D} \vee \mathbf{ABC\bar{D}}$, and no set of $\mathscr{F}'$ contains the element 1. By Lemma 7 we obtain that $\mathscr{F}'$ satisfies $\mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{C}D} \vee \mathbf{ABC\bar{D}}$. On the other hand no pair (X, Y) of $\mathscr{F}'$ can satisfy $\mathbf{AB\bar{C}D}$, i.e. $\mathscr{F}'$ satisfies $\mathbf{A\bar{B}CD} \vee \mathbf{ABC\bar{D}}$ and by **20**:

$$2r-3 \leqslant n(\mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{C}D} \vee \mathbf{ABC\bar{D}}, r) \leqslant n(\mathbf{A\bar{B}CD} \vee \mathbf{ABC\bar{D}}, r) = 2r-3.$$

22. $\mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{C}} \vee \mathbf{AB\bar{D}} \in \mathscr{K}_{10,1}$. By

$$\mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{C}} \vee \mathbf{AB\bar{D}} = (\mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{C}D} \vee \mathbf{ABC\bar{D}}) \vee \mathbf{AB\bar{C}\bar{D}}$$

we get in analogy to **18** that

$$\begin{aligned} n(\mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{C}} \vee \mathbf{AB\bar{D}}, r) &\leqslant 2n(\mathbf{A\bar{B}CD} \vee \mathbf{AB\bar{C}D} \vee \mathbf{ABC\bar{D}}, r) \\ &= 2(2r-3) = 4r-6. \end{aligned}$$

Example 10 completes the proof.

23. $\mathsf{ACD} \vee \mathsf{AB\bar{\bar{C}}\bar{\bar{D}}} \in \mathscr{K}_{5,1}$. The author was not able to determine $f(r) = n(\mathsf{ACD} \vee \mathsf{AB\bar{\bar{C}}\bar{\bar{D}}}, r)$ exactly, but we know the following facts.

(1) $f(r) \geqslant \max(n(\mathsf{ACD}, r), n(\mathsf{ABCD} \vee \mathsf{AB\bar{\bar{C}}\bar{\bar{D}}}, r)) = \begin{cases} \binom{r}{\frac{1}{2}r} & \text{for } r = 2, 4, 6, 8, \\ 2^{r-2} & \text{otherwise.} \end{cases}$

(2) Let $\mathscr{F}_1 \subsetneqq \mathscr{F}$ with $R \setminus X \in \mathscr{F} \setminus \mathscr{F}_1$ for all $X \in \mathscr{F}_1$ and $\mathscr{F} \setminus \mathscr{F}_1$ satisfies ACD. Let $X \in \mathscr{F}$, $R \setminus X \in \mathscr{F}$ and $Y \in \mathscr{F}$. Then (X, Y) and (Y, X) satisfy ABCD. (Without loss of generality let (X, Y) satisfy $\mathsf{A\bar{\bar{B}}CD}$. Then $(R \setminus X, Y)$ satisfies $\mathsf{ABC\bar{\bar{D}}}$ or $(Y, R \setminus X)$ satisfies $\mathsf{AB\bar{\bar{C}}D}$, not $\mathsf{ACD} \vee \mathsf{AB\bar{\bar{C}}\bar{\bar{D}}}$.) Hence $\mathscr{F}_1$ satisfies ABCD and $\mathscr{F} \setminus \mathscr{F}_1$ satisfies ACD. Hence

$$f(r) \leqslant 2^{r-2} + \binom{r-1}{[\frac{1}{2}(r-2)]}.$$

7. Concluding remark

We conjecture

$$f(r) = n(\mathsf{ACD} \vee \mathsf{AB\bar{\bar{C}}\bar{\bar{D}}}, r) = \begin{cases} \binom{r}{\frac{1}{2}r} & \text{for } r = 2, 4, 6, 8, \\ 2^{r-2} & \text{otherwise.} \end{cases}$$

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